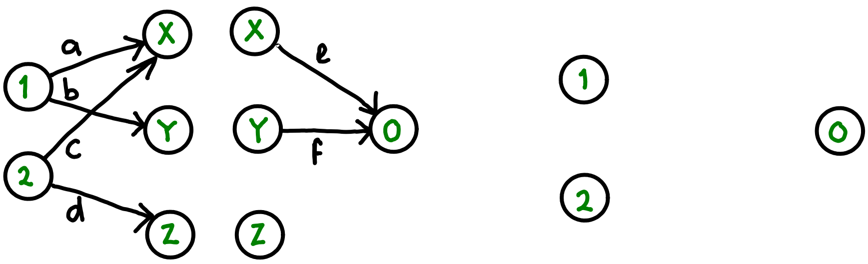
Note Title 21/11/2012

Kleene Algebra

Algebra of choice $(+)$, sequencing $(\cdot)$ and iteration $(*)$.

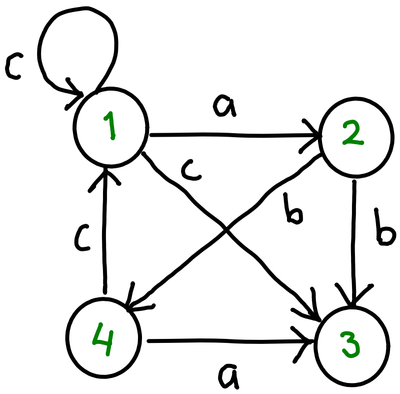
Product of Graphs



Product of Matrices

$$\begin{bmatrix} a & b & 0 \\ c & 0 & d \end{bmatrix} \times \begin{bmatrix} e \\ f \\ 0 \end{bmatrix}$$

Graph



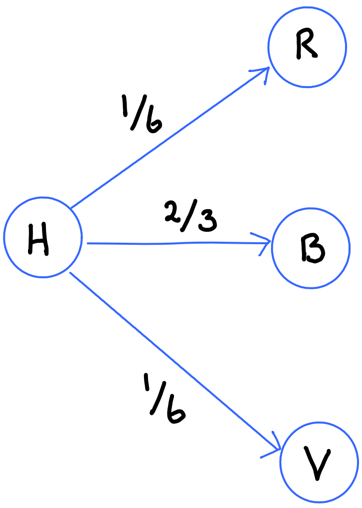
Matrix

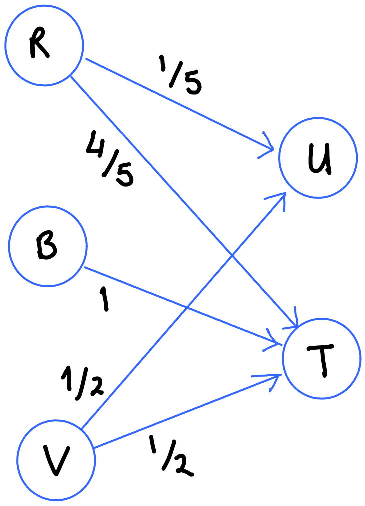
$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} c & a & c & 0 \\ 0 & 0 & b & b \\ 0 & 0 & 0 & 0 \\ c & 0 & a & 0 \end{bmatrix} \end{array}$$

(Homogenous Graph)

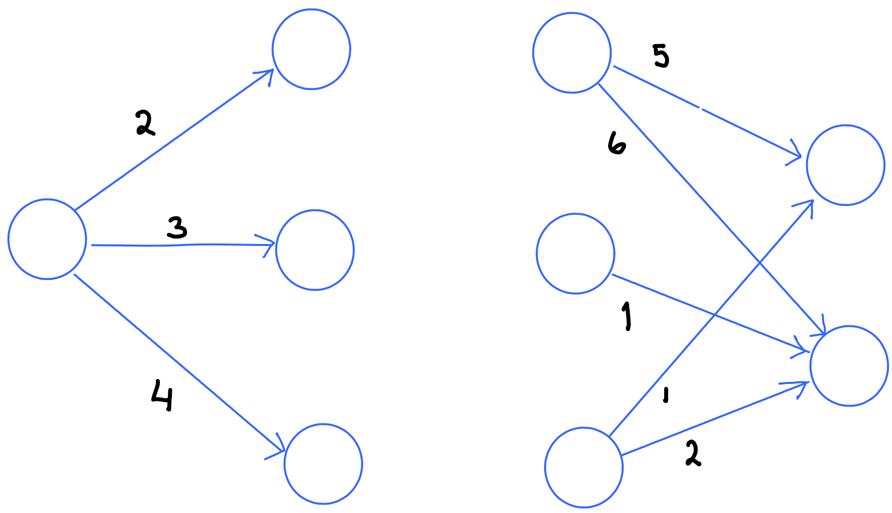
(Square Matrix)

Probabilities

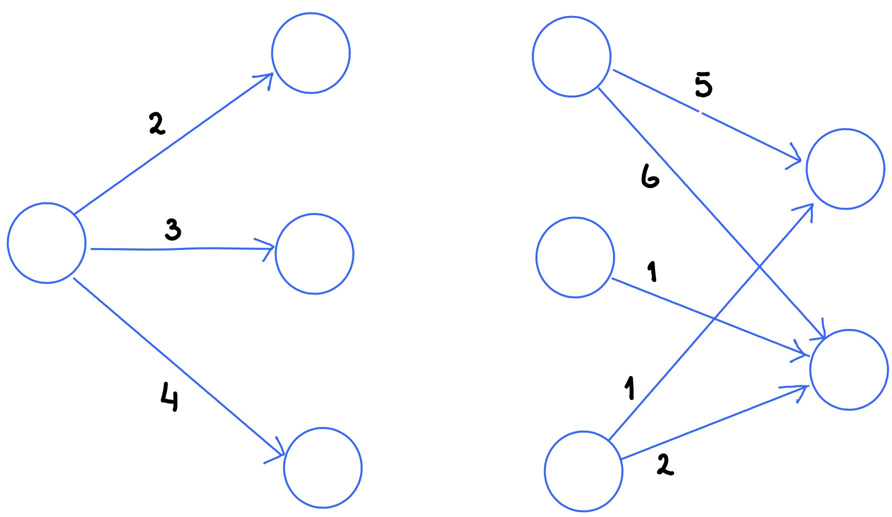




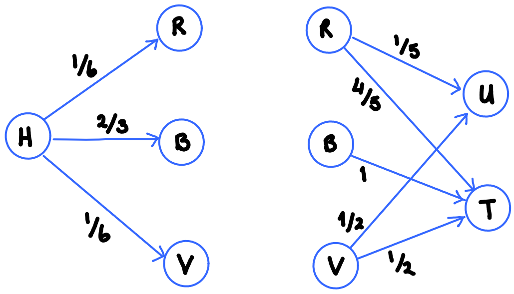
Shortest Distances

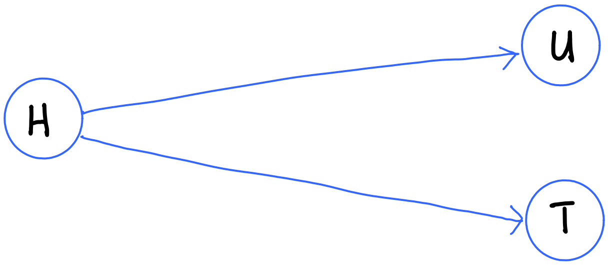


Bottleneck

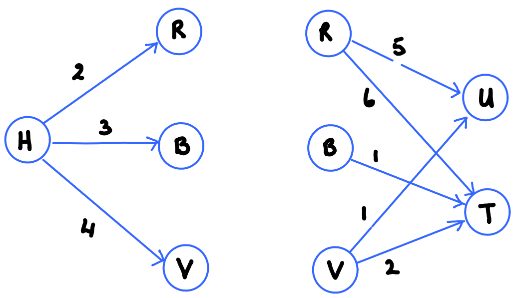


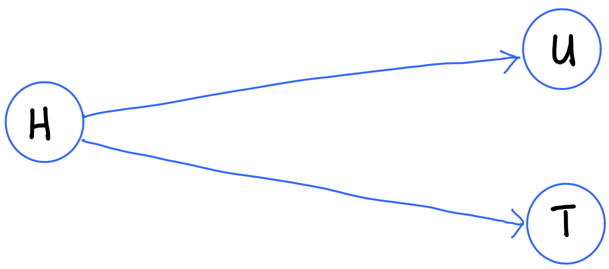
Probabilities



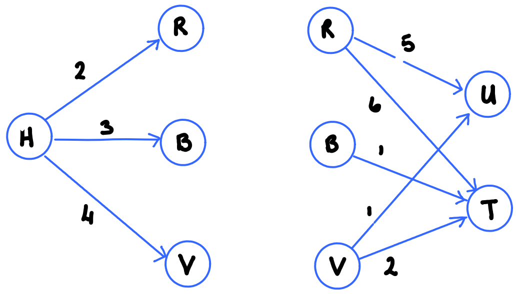


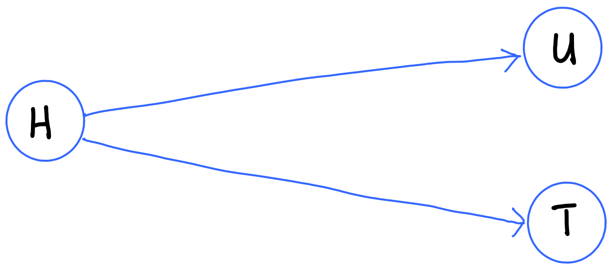
Shortest Distances





Bottleneck





Algebras

A (single-sorted) <u>algebra</u> comprises

- a set, called the <u>carrier</u> of the algebra.
- a collection of <u>operators</u>, each of which has a certain <u>arity</u>. Arities are numbers.
  (0-ary operators are called <u>constants</u>
  1-ary operators are called <u>unary</u>
  2-ary operators are called <u>binary</u>)
- a collection of <u>properties</u>.

Monoid

A monoid is a triple $(M, \times, 1)$ such that

- M is a set (the "carrier" of the monoid),
- $\times$ is a binary operator,
- 1 is a constant,
- $\times$ is associative, i.e. for all $x, y, z \in M$,
  $$x \times (y \times z) = (x \times y) \times z$$
- 1 is the unit of $\times$, i.e. for all $x \in M$,
  $$x \times 1 = x = 1 \times x .$$

Examples

$(\mathbb{N}, \times, 1)$ is a monoid

$(\mathbb{N}, +, 0)$ is a monoid

$(T^*, \cdot, \varepsilon)$ is a monoid

T^* denotes the set of all words over alphabet T

$\cdot$ denotes concatenation of words

ε denotes the empty word

$(\{true, false\}, \equiv, true)$ is a monoid.

$(\{true, false\}, \vee, false)$ is a monoid.

Group

A *group* is a 4-tuple $(G, \times, 1, ^{-1})$ such that

- $(G, \times, 1)$ is a monoid
- $^{-1}$ (called *inverse*) is a unary operator
- For all $x \in G$,

$$x \times x^{-1} = 1 = x^{-1} \times x$$

$$(x^{-1})^{-1} = x \quad .$$

Examples

$(\mathbb{R} - \{0\}, \times, 1, ^{-1})$ is a group

$(\mathbb{Z}, +, 0, -)$ is a group

$(\mathbb{Z}_p, +_p, 0, -_p)$ is a group, if p is a prime

$\mathbb{Z}_p$ denotes $\{0, 1, 2, \ldots, p-1\}$ (numbers mod p)

$+_p$ denotes addition mod p

$-_p$ denotes negation mod p

$(\mathbb{Z}_p, +_p, 0, -_p)$ is not a group, if p is not prime.

Partially Ordered Set (Poset)

A partial ordering is a relation on a set that is reflexive, transitive and antisymmetric.

(Let R be a relation on set A.

R is reflexive if, for all $x \in A$, xRx.

R is transitive if, for all $x, y, z \in R$,

$$xRy \wedge yRz \Rightarrow xRz .$$

R is antisymmetric if, for all $x, y \in A$,

$$xRy \wedge yRx \Rightarrow x = y .)$$

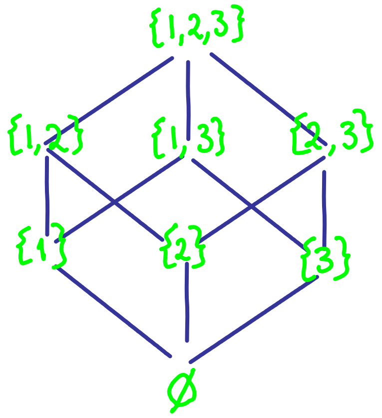
Examples

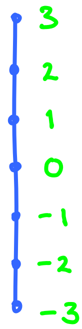
The subset relation, denoted by $\subseteq$, is a partial ordering.

The divides relation, denoted by $\backslash$, is a partial ordering on the natural numbers.

$$(\; m \backslash n \;\; \equiv \;\; \langle \exists k :: n = m \times k \rangle \;)$$

Hasse Diagrams





Subsets of $\{1,2,3\}$ $\qquad\qquad$ $\leqslant$ ordering on integers

(Diagram shows transitive <u>reduction</u> of the relation)

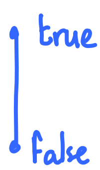
A partial ordering is usually called an <u>ordering</u>.
("partial" is omitted).

An ordering $\precsim$ is <u>total</u> if, for each x and y,

$$x \precsim y \;\vee\; y \precsim x \;.$$

Examples

$\leqslant$ ordering on integers



$\Rightarrow$ ordering on booleans

Semiring

A semiring is a tuple $(S, +, \cdot, 0, 1, \leqslant)$ such that

- $(S, +, 0)$ is a commutative monoid
- $(S, \cdot, 1)$ is a monid
- $[\, x \cdot (y + z) = (x \cdot y) + (x \cdot z) \,]$
- $[\, (x + y) \cdot z = (x \cdot z) + (y \cdot z) \,]$
- $[\, x \leqslant y \equiv x + y = y \,]$
- $\leqslant$ is a partial ordering on S.

Instances

	carrier	$+$	$\cdot$	0	1	$\leqslant$
Languages	sets of words	$\cup$	$\cdot$	$\emptyset$	$\{\varepsilon\}$	$\subseteq$
Programming	binary relations	$\cup$	$\circ$	$\emptyset$	id	$\subseteq$
Reachability	bool	$\vee$	$\wedge$	false	true	$\Rightarrow$
Shortest Paths	nonnegative reals	$\downarrow$	$+$	∞	0	$\geqslant$
Bottlenecks	nonnegative reals	$\uparrow$	$\downarrow$	0	∞	$\leqslant$

Iteration ("Kleene star")

For all a, b,

$$b + a \cdot (a^* \cdot b) \leqslant a^* \cdot b$$
$$b + (b \cdot a^*) \cdot a \leqslant b \cdot a^*$$

For all a, b, x,

$$a^* \cdot b \leqslant x \Leftarrow b + a \cdot x \leqslant x$$
$$b \cdot a^* \leqslant x \Leftarrow b + x \cdot a \leqslant x$$

Properties

Computation $\quad a^* \cdot b = b + a \cdot a^* \cdot b$

$\quad b \cdot a^* = b + b \cdot a^* \cdot a$

Reflexivity $\quad 1 \leqslant a^*$

Transitivity $\quad a^* = a^* \cdot a^*$

Closure Operator $\quad a \leqslant b^* \equiv a^* \leqslant b^*$